Transient Vibration of the single degree of freedom systems.

1. -INTRODUCTION.

Transient vibration is defined as a temporarily sustained vibration of a mechanical system. It may consist of forced or free vibrations, or both (1). Transient loading, also known as impact, or mechanical shock, is a nonperiodic excitation, which is characterized by a sudden and severe application.

In real life, mechanical shock is very common. Some examples of shock could be a forging hammer, an automobile passing trough a road bump, the free drop of an item from a height, etc.

In analysis of systems involving mechanical shock, most of times is necessary to idealize the forcing function (displacement, velocity, acceleration or force) of such system, as a step or pulse function. Several shapes of step and pulse functions are discussed here, as well the response of the system, regarding time history, and frequency response. In first instance, SDOF system without damping are analyzed, then, the effect of damping will be considered.

2. - DEFINITIONS

To avoid confusion, it is necessary to define some useful terminology, which will be used during this discussion.

The spectrum of a given forcing function is a plot of a response quantity, chosen to represent one aspect of the effect of the force function in a single degree of freedom oscillator, against the ratio of the characteristic period or frequency of the forcing function to the natural period or frequency of the oscillator (2). Regarding shock phenomena, it is more convenient to use the period (duration) of the impulse, and the natural period of the system involved, rather than frequencies. This approach is convenient because in transient loading the excitation is relatively short in duration, or has the nature of a single pulse. This lead to use a period spectrum (2).

The maximum absolute displacement, velocity, or acceleration of the system occurring at any time as a result of the forcing function, is called the \textit{maximax response}, denoted by $\nu_{m}$. (2)

The maximum displacement of the system during the residual vibration era, called the residual amplitude, and measured with respect to the final position of equilibrium, denoted by $\nu_{r}$. (2)
3. - SINGLE DEGREE OF FREEDOM SYSTEMS

3.1. - Undamped, linear system.

Let's consider a SDOF system, linear, and with no damping. The input is a function of time, and may be a force function acting on the mass, or a displacement of the base or foundation. Sometimes is more convenient to express it as ground acceleration. (Fig. 1)

\[ \ddot{x} = -kx + F(t) \quad \text{or} \quad \frac{m\ddot{x}}{k} + x = \frac{F(t)}{k} \quad (1) \]

\[ \ddot{x} = -k[x - u(t)] \quad \text{or} \quad \frac{m\ddot{x}}{k} + x = u(t) \quad (2) \]

\[ m \left[ \ddot{x} + x(t) \right] = -k\delta_x \quad \text{or} \quad \frac{m\ddot{x}}{k} + \delta_x = -\frac{m\ddot{u}(t)}{k} \quad (3) \]

Where \( x \) is the absolute displacement of the mass relative to a fixed reference,
and $\delta_x$ is the displacement relative to a moving ground. The relationship between these displacements and the ground displacement is $x = u + \delta_x$.

In this report, the motion of a system will be described using a more general form of the equations (1, 2, 3). The equation is:

$$\frac{m}{k} \ddot{v} + v = \xi(t) \quad (4)$$

Where $v$ is the response of the system, and $\xi$ the excitation, both functions of time. However, sometimes it is necessary to express excitation and response using more specific notations. The alternate forms of excitation and response are given in table 1(1).

<table>
<thead>
<tr>
<th>Excitation $\xi(t)$</th>
<th>Response $v(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force $\frac{F(t)}{k}$</td>
<td>Absolute displacement $x$</td>
</tr>
<tr>
<td>Ground displacement $u(t)$</td>
<td>Absolute displacement $x$</td>
</tr>
<tr>
<td>Ground acceleration $\frac{-u(t)}{\omega_n^2}$</td>
<td>Relative displacement $\delta_x$</td>
</tr>
<tr>
<td>Ground acceleration $u''(t)$</td>
<td>Absolute acceleration $\ddot{x}$</td>
</tr>
<tr>
<td>Ground velocity $u'(t)$</td>
<td>Absolute velocity $\dot{x}$</td>
</tr>
<tr>
<td>$n$th derivative of ground displacement $\frac{d^n u}{dt^n}(t)$</td>
<td>$n$th derivative of absolute displacement $\frac{d^n x}{dt^n}$</td>
</tr>
</tbody>
</table>

Table 1. - Alternate forms of excitation and response of equation 4.
3.1.2. - Steplike and pulselike excitation.

The impulsive excitations cause vibrational responses in elastic systems, and the maximum values of these responses may be less than, equal to, or greater than the corresponding static responses (3). In general, the response depends upon the system properties, and the nature of the load. For single degree of freedom systems, the characteristic that determines the response is the natural period (or natural frequency). In addition, the shape and duration of the impulse plays an important role in the response. Shock phenomena can be modeled using ideal step and pulse functions, which represent very well the behavior of real transient inputs.

The following analysis is based on the assumption that the system is initially at rest.

**Step type excitation functions.**

The most fundamental transient excitation is the form of the step function. To be fully realistic, these functions must describe the translation of the system through a finite distance, in a finite time, with finite acceleration and deceleration (4). Many functions rise to their maximum \( \xi_c \) in a finite time \( t \), called *rise time*. Consider the three next functions, the excitation functions and the expressions for maximax response are given by (1):

a) Constant slope front.

\[
\xi(t) = \begin{cases} 
\xi_c \frac{t}{\tau} & [0 \leq t \leq \tau] \\
\xi_c & [\tau \leq t] 
\end{cases}
\]  

(5)

\[
\nu_m = 1 + \left| \frac{T}{\pi \tau} \sin \left( \frac{\pi \tau}{T} \right) \right| 
\]  

(6)

a) Versed sine front front.

\[
\xi(t) = \begin{cases} 
\frac{\xi_c}{2} \left( 1 - \cos \left( \frac{\pi t}{\tau} \right) \right) & [0 \leq t \leq \tau] \\
\xi_c & [\tau \leq t] 
\end{cases}
\]  

(7)
\[
\frac{V_m}{\xi_c} = 1 + \frac{1}{\left(4\tau^2/T^2\right) - 1} \cos\left(\frac{\pi \tau}{T}\right)
\]

(7)

a) Cycloidal front.

\[
\xi(t) = \begin{cases} 
\frac{\xi_c}{2} & \frac{2\pi t}{\tau} - \sin\left(\frac{2\pi t}{\tau}\right) \\
\xi_c & \tau \leq t 
\end{cases} \quad [0 \leq t \leq \tau]
\]

(8)

\[
\frac{V_m}{\xi_c} = 1 + \frac{1}{\pi \tau\left(1 - \tau^2/T^2\right)} \sin\left(\frac{\pi \tau}{T}\right)
\]

(9)

Where \( T \) is the natural period of the system involved.

The plots for the excitation function, and the time response curves are superimposed in fig. 2.

Figure 3.- Time response curves and excitation functions for (a) constant slope front, (b) versed sine front and (c) cycloidal front.

Symmetrical pulses.

A pulselike excitation is a more complex function, being equivalent to the superposition of two or more successive step functions. Consider three single symmetrical pulses; rectangular, half sine and versed sine. The excitation functions and time response equations are given by the following equations (1). Residual response factors are set in brackets.
a) Rectangular

\[
\begin{align*}
\dot{\xi}(t) &= \xi_p \\
\nu &= \xi_p \left(1 - \cos(\omega_n t)\right) \\
\end{align*}
\]  
\[0 \leq t \leq \tau\]  \(10\)

\[
\begin{align*}
\dot{\xi}(t) &= 0 \\
\nu &= \xi_p \left[2 \sin\left(\frac{\pi t}{T}\right)\right] \sin\omega_n\left(t - \frac{\tau}{2}\right) \\
\end{align*}
\]  
\[\tau \leq t\]  \(11\)

b) Half cycle sine

\[
\begin{align*}
\dot{\xi}(t) &= \xi_p \sin\left(\frac{\pi t}{\tau}\right) \\
\nu &= \frac{\xi_p}{1 - T^2/4\tau^2} \sin\left(\frac{\pi t}{\tau}\right) - \frac{T}{2\tau} \sin(\omega_n t) \\
\end{align*}
\]  
\[0 \leq t \leq \tau\]  \(12\)

\[
\begin{align*}
\dot{\xi}(t) &= 0 \\
\nu &= \xi_p \left[\frac{(T/\tau)\cos(\pi T/\tau)}{(T^2/4\tau^2) - 1}\right] \sin\omega_n\left(t - \frac{\tau}{2}\right) \\
\end{align*}
\]  
\[\tau \leq t\]  \(13\)

c) Versed sine

\[
\begin{align*}
\dot{\xi}(t) &= \frac{\xi_p}{2} \left(1 - \cos\left(\frac{2\pi t}{\tau}\right)\right) \\
\nu &= \frac{\xi_p}{1 - \tau^2/T^2} \left(1 - \frac{\tau^2}{T^2} + \frac{\tau^2}{T^2}\cos\left(\frac{2\pi t}{\tau}\right) - \cos(\omega_n t)\right) \\
\end{align*}
\]  
\[0 \leq t \leq \tau\]  \(14\)

\[
\begin{align*}
\dot{\xi}(t) &= 0 \\
\nu &= \xi_p \left[\frac{\sin(\pi T/\tau)}{1 - \tau^2/T^2}\right] \sin\omega_n\left(t - \frac{\tau}{2}\right) \\
\end{align*}
\]  
\[\tau \leq t\]  \(15\)

Time response curves and excitation functions are shown in fig. 5, for different values of the ratio \(\tau/T\).
Figure 5.- Time response curves for several symmetrical pulses; rectangular, half sine, and versed sine, for different values of $\tau/T$. 

<table>
<thead>
<tr>
<th>Ratio $\tau/T$</th>
<th>Rectangular</th>
<th>Half sine</th>
<th>Versed sine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
<td><img src="image3" alt="Graph" /></td>
</tr>
<tr>
<td>1/2</td>
<td><img src="image4" alt="Graph" /></td>
<td><img src="image5" alt="Graph" /></td>
<td><img src="image6" alt="Graph" /></td>
</tr>
<tr>
<td>1</td>
<td><img src="image7" alt="Graph" /></td>
<td><img src="image8" alt="Graph" /></td>
<td><img src="image9" alt="Graph" /></td>
</tr>
<tr>
<td>3/2</td>
<td><img src="image10" alt="Graph" /></td>
<td><img src="image11" alt="Graph" /></td>
<td><img src="image12" alt="Graph" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image13" alt="Graph" /></td>
<td><img src="image14" alt="Graph" /></td>
<td><img src="image15" alt="Graph" /></td>
</tr>
<tr>
<td>5/2</td>
<td><img src="image16" alt="Graph" /></td>
<td><img src="image17" alt="Graph" /></td>
<td><img src="image18" alt="Graph" /></td>
</tr>
</tbody>
</table>
3.2. - Shock spectrum.

Any mechanical system is capable of transient vibrations in the form of motions of one part with respect to another, and involving resonances at a number of frequencies (5). The primary effect of shock is to excite the equipment at its resonance frequencies. This vibration dies away depending on the amount of damping present in the system. As a result, damage or malfunction may take place. The peak acceleration and peak relative displacement of the system are particularly important.

Spectrum was defined as a plot of a response quantity (displacement, velocity or acceleration), against the ratio of the period or frequency of the forcing function, to the natural period or frequency of the system. In this report, we will refer to shock spectra, also regarded as response spectra. Shock spectra are simply the peak acceleration or displacement produced by the shock in the mass as a function of the resonant frequency (4). Shock spectrum can be experimentally measured, computed from waveforms, or determined theoretically. Such diagrams are of interest in design, because they provide the possibility of predicting the maximum dynamic stress (3), and the potential damage in the system.

In fact, the shock spectrum gives a full and realistic measure of the damaging potential of a shock disturbance. To select a damage criterion (acceleration or displacement), the duration of the pulse and the natural period of the system are of great importance. If $T \geq \tau$, the system behavior becomes stiff, and the motion of the mass closely follows the motion of the support. The acceleration becomes the quantity of concern. Otherwise, when $T < \tau$, the element becomes soft, and the mass remains substantially at rest until motion of the support has been ceased, is the maximum displacement the value that determines potential damage. Moreover, if the transient disturbance is neither fast nor slow enough to fit into one of the previous cases, no simple damage criterion can be found (4).

3.2.1 - Shock spectra for particular pulse and step shapes.

Further insight into the significance of shock spectrum can be obtained by studying the spectra corresponding to a number of simply shapes (5). Consider the functions discussed on section 3.2.1 and 3.2.2.

Fig. 6 shows the spectra for maximax response resulting from the step functions discussed on section 3.2.1, plotted as a function of the ratio between rise time and natural period of the system. Maximax response occurs after the excitation as reached its constant maximum, and is related to the residual amplitude by (1):

$$v_{\text{max}} = v_{r} + \xi$$

(16)
From the analysis of these graphs we can observe that the extreme values of the ratio of maximum response to step height $\nu_m/\xi_c$ are 1 and 2. When the ratio of step rise time to natural period $\tau/T$ tends to 0, $\nu_m/\xi_c$ tends to its maximum of 2, and when $\tau/T$ approaches infinity, the step is no longer a dynamic excitation, in consequence the inertia forces of the system tends to zero, and $\nu_m/\xi_c$ approaches the lower value of 1(1).

It is clear that for certain values of $\tau/T$, the ratio $\nu_m/\xi_c$ is equal to 1. The lowest value of $\nu_m/\xi_c$ for which $\nu_m/\xi_c = 1$, are: constant slope, 1.0; versed sine, 1.5; cycloidal, 2.0 (1).

Response spectra for symmetrical pulses (rectangular, half-sine and versed sine) are shown in fig. 6. A further analysis of these plots and the time response curves (Fig. 5) reveals that for values of $\tau/T$ less than $\frac{\sqrt{2}}{4}$ (this is the case of short duration pulses), the shape of the pulse is of less importance in determining the maximum value of the response, but, in contrast, if $\tau/T$ is larger than $\frac{\sqrt{2}}{2}$, the shape may be of great significance.

The maximum value of the residual response amplitude for the shapes discussed is often a good approximation to maximum of maximax response, and they occur at values of $\tau/T$ not greatly different from each other (1). The residual response amplitude generally has zero values for certain finite values of $\tau/T$ (1).
Figure 6.- Shock response spectrum for maximax and residual amplitude for three symmetrical pulses; (a) rectangular, (b) half sine, (c) versed sine.
3.2. - Linear system with viscous damping.

Regarding steady forced vibration, even if the system has little values of damping, it has great importance in limiting the system response near resonance. However, if the excitation is a pulse or step function, the effect of damping may be of less importance, unless the system is highly damped (1). Nevertheless, as a result of the introduction of damping in the system, maximax response decreases, as well the residual response. Although, less work had been done in damped systems under shock excitations.

For damped systems, the equation of motion under general notation, is:

$$\frac{m^{\dddot{v}}}{k} + \frac{c^{\dot{v}}}{k} + v = \ddot{\xi}(t)$$

or

$$\frac{1}{\omega_n^2} \dddot{v} + \frac{2\zeta}{\omega_n} \dot{v} + v = \ddot{\xi}(t)$$

Figure 6 shows a response spectrum for a single degree of freedom system with viscous damping, subjected to a half sine pulse excitation (1).

Figure 6. - Maximax response spectrum for a single degree of freedom system with viscous damping, subjected to a half sine pulse excitation, for various values of damping ratio(1).

3. - CONCLUSIONS AND REMARKS.

The most important quantities in shock measurements are the maximax response and the residual response, both used to determine the severity of the shock. A powerful tool in the analysis of shock motions is the shock response spectrum, which gives information about the relationship between maximax response, and the duration of the shock and the natural period of the system.
From the shock response spectrum and time responses for step functions, it is clear that the maximax response occurs always after the excitation has reached its maximum. On the other hand, for pulse functions, the response depends upon the ratio between the duration of the pulse and the natural period of the system. Or, in other words, if the pulse is of short duration, the shape of the pulse has not importance (for $\tau/T$ less than $1/2$), and the maximax occurs after the excitation has ceased. In contrast, when the pulse is long, the shape plays an important role, and maximax response can occur during the pulse, and after it has ceased. For some values of $\tau/T$, residual response its zero.

The inclusion of viscous damping in the system causes a decrease in the maximax response. Most of the work has been carried out considering only viscous damping. Further research can be done in investigating the behavior of the system when the damping is structural.

3. - REFERENCES.

2. Ayre, R. S.: Engineering vibrations, 1958
3. Timoshenko S, Young D H, Weaver W.: Vibration problems in engineering, 1974